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THE REVERSION OF CLASS NUMBER RELATIONS AND THE TOTAL REPRESENTATION OF INTEGERS AS SUMS OF SQUARES OR TRIANGULAR NUMBERS.

By E. T. Bell.

We shall discuss a set of new arithmetical functions defined in §§ 7, 8 relating to representations as sums of square or triangular numbers, the connection of these with class numbers, and means for calculating by recurrence the numerical values of the functions. The new functions first present themselves in reversing the class number formulas of the classical types due to Kronecker, Hermite and Liouville. They are themselves connected by many relations of a like simplicity, and seem to deserve attention on their own account. In section I we fix the notation and state the sense in which reversion is used throughout; in II the functions are defined and their generating series determined, the absolute convergence of these being proved incidentally; III contains four examples of the reversion of simple class number relations, and IV gives a short selection from the numerous recurrences between the functions, those chosen for presentation being among the most useful for numerical computations.

I. Notation; Reversions.

In the customary notation let $F(n)$, $F_1(n)$ denote the number of odd, of even classes respectively of binary quadratic forms for the determinant $-n$, so that $G(n) = F_1(n) + F(n)$ is the whole number of classes, and write

$$H(n) = F(n) - F_1(n).$$

By the usual conventions a class equivalent to $a(x^2 + y^2)$ contributes ½ to F or F_1; one equivalent to $a(2x^2 + 2xy + 2y^2)$ counts for $1/3$ in F_1. It is simpler in the sequel to ignore the other conventions $F(0) = 0$, $F_1(0) = -1/12$; hence all formulas involving $F(n)$, $F_1(n)$ or other arithmetical functions will be so stated as to preclude the occurrence of zero values of the argument n.

Henceforth, without further references, $(\alpha \mid \beta)$ is the Jacobi-Legendre symbol; m, μ, n, a, b, t are integers > 0, of which m, μ are odd, n, a, b arbitrary, t is triangular ($= 1, 3, 6, 10, \cdots$), and k is an integer $\geqq 0$. In all power series in q, in particular in those for the elliptic theta constants $\vartheta_a = \vartheta_a(q)$, $\vartheta_1' = \vartheta_0\vartheta_2\vartheta_3$, $\vartheta_0 = 1 + 2\Sigma(-1)^n q^{n^2}$, $\vartheta_2(q^4) = 2\Sigma q^{m^2}$, $\vartheta_3 = 1$

$+ 2\Sigma q^{n^2}$, $\vartheta_1'(q^4) = 2\Sigma(-1|m)mq^{m^2}$, the summations refer to all values from 1 to ∞ of the exponents consistent with the m, n notation. But in all sums independent of q, such as $\Sigma f(2m - a^2)$, the Σ is with respect to the letters μ, a, b, or t involved, and extends only to all those values (of the μ, a, b, or t) that make the argument > 0, so that any such sum consists of only a finite number of terms and zero values of the argument do not occur. When in any sum any of the integers are restricted beyond the notation already explained the restrictions will be given explicitly. Thus

$$m = 8k + 3 : \Sigma f(2m - a^2) = 0$$

indicates that the sum vanishes only when $m \equiv 3 \bmod 8$.

2. A function $f(x)$ which takes a single definite value when x is an integer > 0 is called arithmetical. Let α, β denote arithmetical functions between which there is the relation

$$\alpha(n) = (-1)^n \begin{vmatrix} \beta(1) & \beta(2) & \beta(3) & \cdots & \beta(n) \\ 1 & \beta(1) & \beta(2) & \cdots & \beta(n-1) \\ 0 & 1 & \beta(1) & \cdots & \beta(n-2) \\ 0 & 0 & 1 & \cdots & \beta(n-3) \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & \beta(2) \\ 0 & 0 & 0 & \cdots & \beta(1) \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix} . \tag{1}$$

We shall call $\alpha(n)$ the inverse of $\beta(n)$, or simply α the inverse of β, a reason for this nomenclature appearing in a moment. On expanding the determinant by minors of the elements in its last column, we see that (1) is equivalent to

$$\alpha(n) + \beta(n) + \Sigma\alpha(a)\beta(n - a) = 0; \tag{2}$$

and this being symmetric in α, β, it follows that if α is the inverse of β, then β is the inverse of α.

3. The problem of reversing class number relations is presently reduced to finding the inverses of certain elementary arithmetical functions. When α is arithmetically defined it is not always easy à priori to give an explicit arithmetical definition of its inverse β. Thus if $H'(n)$ is the inverse of $12H(n)$ defined in § 1, it may be verified from Dirichlet's formulas for the class number combined with Gauss' theorems on decompositions into sums of three squares that

$$H'(n) = \sum_{r=1}^{n}(-1)^r t_{r+1} N_r'(n),$$

in which t_a denotes the ath triangular number and $N_r'(n)$ is the total number of representations of n as a sum of r squares whose roots are $\gtreqless 0$. But the verification is artificial and involved, and all such questions are better treated by direct algebraic methods, one of which is elaborated in this paper in detail sufficient for the reversions of class number formulas of any of the classical types.

4. Consider three pairs of arithmetical functions, (P, P'), (Q, Q'), (R, R'), the functions in any pair being inverses of each other, and let P, Q, R be connected by the relation

$$P(n) + Q(n) + \Sigma P(a)Q(n - a) = R(n), \tag{3}$$

which may conveniently be symbolized by PQR in which, note, the function given explicitly in terms of the other two occurs last. The process of solving PQR for P is called the reversion of PQR with respect to P, and the solution the P-reverse of (3). We will show that

$$R(n) + Q'(n) + \Sigma Q'(a)R(n - a) = P(n); \tag{4}$$

that is, the P-reverse of PQR is $RQ'P$, or what is the same thing, by symmetry, $Q'RP$. Hence we have the rule: To reverse any relation of the form PQR with respect to either function given implicitly, interchange the function given explicitly and the function with respect to which the reversion is taken, and replace the other function by its inverse.

It is easily seen that (3) implies (4), viz., that PQR implies $RQ'P$. For if in (4) we replace $R(n)$, $R(n - a)$ by their values as given by (3), the latter being

$$R(n - a) = P(n - a) + Q(n - a) + \sum_b P(b)Q(n - a - b),$$

and collect coefficients of $P(a)$, we find

$$\begin{aligned}&[Q(n) + Q'(n) + \Sigma Q'(a)Q(n - a)] \\ &\qquad + \sum_a [Q(n - a) + Q'(n - a) + \sum_b Q'(b)Q(n - a - b)]P(a) = 0,\end{aligned}$$

which is an identity, each square bracket vanishing separately since Q, Q' are inverses.

We have just shown that PQR implies $RQ'P$. From this it follows, since if Q' is the inverse of Q then Q is the inverse of Q', that $RQ'P$ implies PQR. Hence in the meaning of mathematical logic PQR and $RQ'P$ are formally equivalent, $PQR \equiv RQ'P$.

It is evident that from any relation of the type PQR we can by reversions obtain six and only six relations of the same type,

$$Q'RP, \qquad P'RQ, \qquad PQR, \qquad QR'P', \qquad PR'Q', \qquad P'Q'R'.$$

These determine, in the same order, P, Q, R, P', Q', R' from the given relation PQR; and clearly from what precedes, any two of the six are formally equivalent and each implies all.

A relation of type PQR is thus six-valued, and the six values constitute its complete reversion. When considering the reversion of class number relations we shall confine the discussion to the partial reversions which give the class number functions explicitly in terms of known functions and their inverses.

5. We need also the reverses of another type of relation, (PQR), viz.,

$$P(n) + \Sigma P(a)Q(n - a) = R(n). \tag{6}$$

As before it is seen at once that this is two-valued, and the complete reversion is

$$(PQR), \qquad (RQ'P). \tag{7}$$

6. If for all values of q defined by $0 < |q| < c$ where c is a constant, the series

$$\gamma(f) = 1 + \Sigma q^n f(n)$$

converges absolutely, $\gamma(f)$ is called the generator of f. Let f, f' be inverses, and suppose that for the same q both $\gamma(f)$ and $\gamma(f')$ are absolutely convergent. Then from (2) we have, on collecting coefficients of q^n,

$$\gamma(f)\gamma(f') = 1.$$

Hence if for the same q the generators of a function and its inverse are absolutely convergent, the generator of the inverse is the reciprocal of the generator of the function.

Suppose $\gamma(P)$, $\gamma(P')$, $\gamma(Q)$, $\gamma(Q')$, $\gamma(R)$, $\gamma(R')$ are absolutely convergent for the same q, the functions being those in § 4, and suppose further that $\gamma(P)\gamma(Q) = \gamma(R)$. On equating coefficients of q^n in this we find PQR of § 4. Multiplying the identity between the generators throughout by $\gamma(Q')$ we get $\gamma(R)\gamma(Q') = \gamma(P)$, which yields the relation $RQ'P$, viz., the P-reverse of PQR. In this way we find by the appropriate multiplications all six of the relations in the complete reversion of PQR, and similarly for (PQR).

As all of the generators giving rise to class number relations, likewise all of those for the inverses of the several functions occurring in these are absolutely convergent for the same q (see § 9), we shall use the method of generators exclusively in finding the reversions. This method, when it can be applied, is preferable to a direct use of (4), (7) as even in simple cases the necessary arithmetical reductions for the latter are not always apparent.

II. TOTAL FUNCTIONS AND THEIR GENERATORS.

7. The functions defined in § 8 may be regarded as a natural extension of certain functions occurring in the theory of partitions, as they relate to the total number of ways in which an integer may be written as a sum of square or triangular numbers of preassigned forms. By the total number of ways in which n may be written as a sum of squares we mean the sum of the number of ways in which n may be represented as a sum of r squares whose roots are $\gtrless 0$, for $r = 1, 2, \cdots, n$, the order of the squares in any representation being essential. Similarly for the other total functions; *all* the representations of the kinds specified are to be counted, and in each case only squares whose roots are different from zero, or positive triangular numbers, are enumerated in any representation.

It will be noticed in the following functions that the suffix 1 or 2 is of the same parity as the total numbers of odd squares occurring in the several representations of the kinds even (E), or odd (O, Ω), with a similar device for the triangular T, so that the meanings and elementary properties of all the symbols are easily retained.

8. Let $E_1(n)$, $E_2(n)$, $\cdots$, denote the total numbers of representations of n as sums of the following kinds:

(8) $E_1(n)$: even number of squares, the total number of odd squares in each of the representations enumerated being odd; $E_1(2n) = 0$.

(9) $E_2(n)$: even number of squares, the total number of odd squares in each of the representations enumerated being even; $E_2(m) = 0$.

(10) $E(n)$: even number of squares; $E(n) = E_1(n) + E_2(n)$.

(11) $O_1(n)$: odd number of squares, the total number of odd squares in each of the representations enumerated being odd; $O_1(2n) = 0$.

(12) $O_2(n)$: odd number of squares, the total number of odd squares in each of the representations enumerated being even; $O_2(m) = 0$.

(13) $O(n)$: odd number of squares; $O(n) = O_1(n) + O_2(n)$.

(14) $N(n)$: squares; $N(n) = E(n) + O(n)$.

(15) $\Omega_1(n)$: odd number of odd squares; $\Omega_1(2n) = 0$.

(16) $\Omega_2(n)$: even number of odd squares; $\Omega_2(m) = 0$.

(17) $\Omega(n)$: odd squares; $\Omega(n) = \Omega_1(n) + \Omega_2(n)$.

(18) $T_1(n)$: odd number of triangular numbers.

(19) $T_2(n)$: even number of triangular numbers.

(20) $T(n)$: triangular numbers; $T(n) = T_1(n) + T_2(n)$.

(21) $\Phi'(n) = \Phi_2(n) - \Phi_1(n)$, $\Phi = E, O, \Omega, T$.

(22) $D(n) = E(n) - O(n)$; $D'(n) = E'(n) - O'(n)$.

Since $E_1(2n) = 0$, $E_2(2n) = E(2n)$, etc., it may seem that E_1, E_2, O_1, O_2, Ω_1, Ω_2 are superfluous, and that E, O, Ω only are necessary. This of course is true. Nevertheless the statement of many processes is much simplified by retaining all, which we shall do, using one set or the other as convenient. From the definitions we have the useful identities

$$(23) \qquad (-1)^n \Psi(n) = \Psi'(n), \qquad \Psi = E,\ O,\ \Omega,\ D.$$

The functions (8)–(22) are those most frequently required in class number reversions.

9. Let φ denote any one of the functions defined in (8)–(22) except those involving T, T_1 or T_2. Then obviously $|N(n)| \geqq |\varphi(n)|$. Hence the absolute convergency of $\gamma(N)$ for $0 < |q| < c$ implies that of $\gamma(\varphi)$ for the same q. We shall prove that $c = \frac{1}{4}$ ensures the absolute convergency of $\gamma(N)$. This value also makes each of ϑ_α, ϑ_1', and hence also their positive integral powers $\vartheta_\alpha{}^a$ and products $\vartheta_\alpha{}^a \vartheta_\beta{}^b$, etc., absolutely convergent. A larger c $(= 1)$ may be found making $\gamma(N)$ absolutely convergent; but as this is needed in nothing that follows, and as the proof is longer, we omit consideration of this point.

Let $N_r(n)$ denote the total number of representations of n as a sum of r squares whose roots are > 0, and $N_r'(n)$ the total number of representations of n as a sum of r squares whose roots are $\gtrless 0$. Then $N_r'(n) = 2^r N_r(n)$;

$$N(n) = \sum_{r=1}^{n} N_r'(n) = \sum_{r=1}^{n} 2^r N_r(n) \leqq 2^n \sum_{r=1}^{n} N_r(n).$$

Hence $N(n)/2^n \leqq$ the total number of ways in which n may be written as a sum of squares whose roots are > 0. But clearly this last number $\leqq 2^{n-1}$, for 2^{n-1} is the total number of ways into which n may be partitioned into n or fewer positive non-zero integers. Hence $N(n) \leqq 2^{2n-1}$; and therefore since $1 + \Sigma q^n 2^{2n-1}$ converges absolutely if $0 < |q| < \frac{1}{4}$, the absolute convergence of $\gamma(\varphi)$ for the same range is established. And it is obvious that by a few slight changes this argument can be modified to fit those functions of (18)–(20) which involve T, T_1, T_2. Henceforth this value of q is assumed in all the series.

10. Expanding $\vartheta_3{}^{-1}$ in powers of $\vartheta_3 - 1$,

$$\vartheta_3{}^{-1} = [1 + (\vartheta_3 - 1)]^{-1} = 1 + \Sigma(-1)^n (\vartheta_3 - 1)^n,$$

we see at once that the coefficient of q^n is $D(n)$. Thus we have the first fundamental generator,

$$(24) \qquad \gamma(D) = 1/\vartheta_3.$$

Change q into $-q$ and apply (23):

$$(25) \qquad \gamma(D') = 1/\vartheta_0.$$

Similarly from $[1 - (\vartheta_3 - 1)]^{-1}$ we derive the second fundamental series

$$(26) \qquad \gamma(N) = 1/(2 - \vartheta_3);$$

whence, as before,

$$(27)\qquad 1 + \Sigma q^n[E'(n) + O'(n)] = 1/(2 - \vartheta_0).$$

From (24), (26) by addition and subtraction,

$$(28)\qquad \gamma(E) = 1/(2\vartheta_3 - \vartheta_3^2),$$
$$(29)\qquad \gamma(O) - 1 = (\vartheta_3 - 1)/(2\vartheta_3 - \vartheta_3^2);$$

and from these on replacing q by $-q$, or independently from (25), (27),

$$(30)\qquad \gamma(E') = 1/(2\vartheta_0 - \vartheta_0^2),$$
$$(31)\qquad \gamma(O') - 1 = (\vartheta_0 - 1)/(2\vartheta_0 - \vartheta_0^2).$$

By combining (28)–(31) by addition and subtraction we find $\gamma(\Phi) - 1$ for $\Phi = E_1, E_2, O_1, O_2$. The results, which reduce to comparatively simple forms on factoring numerators and denominators, need not be written out here, as they are required in nothing that follows.

The third fundamental series generates T'; and as before the following are readily seen:

$$(32)\qquad 1 + \Sigma q^{8n}T'(n) = 2q/\vartheta_2(q^4);$$
$$(33)\qquad 1 + \Sigma q^{8n}T(n) = 2q/[4q - \vartheta_2(q^4)];$$
$$(34)\qquad 1 + \Sigma q^{8n}T_2(n) = 4q^2/\vartheta_2(q^4)[4q - \vartheta_2(q^4)];$$
$$(35)\qquad \Sigma q^{8n}T_1(n) = q[2\vartheta_2(q^4) - 4q]/\vartheta_2(q^4)[4q - \vartheta_2(q^4)];$$

whence, replacing q by $\sqrt[8]{q}$ we have at once

$$\gamma(T'),\qquad \gamma(T),\qquad \gamma(T_2),\qquad \gamma(T_1) - 1.$$

The fourth set is for Ω:

$$(36)\qquad 1 + \Sigma q^n\Omega(n) = 1/[1 - \vartheta_2(q^4)];$$
$$(37)\qquad 1 + \Sigma q^n\Omega'(n) = 1/[1 + \vartheta_2(q^4)];$$
$$(38)\qquad 1 + \Sigma q^n\Omega_2(2n) = 1/[1 - \vartheta_2^2(q^2)];$$
$$(39)\qquad \Sigma q^m\Omega_1(m) = \vartheta_2(q^4)/[1 - \vartheta_2^2(q^4)].$$

For convenience in numerical checks there is a short table at the end of the paper. All formulas from now on have been checked by means of the table, which was calculated independently. Note that our class number functions will not all agree with those read off from Cayley's table,* as he does not adopt the conventions of § 1. The values in this paper are those which will check in the class number relations of Kronecker, Hermite, Liouville and Humbert, the final form of Kronecker's being that followed.

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* Collected Papers, vol. 5, p. 141.

III. Reversions of Class Number Recurrences.

11. The range of possibilities being very extensive we shall discuss only the four class number relations arising from Hermite's (or Kronecker's) developments* of ϑ_3^3, ϑ_2^3, $\vartheta_2\vartheta_3^2$, $\vartheta_2^2\vartheta_3$.

To state the relations we require the functions ζ, ζ', e, λ: $\zeta(n) =$ the sum of all the divisors of n; $\zeta'(n) =$ the sum of the odd divisors of n, $\zeta'(m) = \zeta(m)$; $e(n) = 1$ or 0 according as n is or is not the square of an integer > 0;

$$\lambda(n) = [1 + 2(-1)^n]\zeta'(n); \qquad \therefore \lambda(m) = -\zeta(m), \qquad \lambda(2n) = 3\zeta'(n).$$

From the developments of the elliptic constants in the Fundamenta Nova, or from the theorems on representations of integers as sums of four squares, we have

$$(40)\qquad \vartheta_0^4 = 1 + 8\Sigma q^n\lambda(n), \qquad \vartheta_2^4 = 16\Sigma q^m\zeta(m);$$
$$(41)\qquad \vartheta_2^2(q^2)\vartheta_3^2(q^2) = 4\Sigma q^m\zeta(m), \qquad \vartheta_0^2\vartheta_3^2 = 1 + 8\Sigma q^{2n}\lambda(n);$$

and Hermite's series are

$$(42)\qquad \vartheta_3^3 = 1 + 12\Sigma q^n H(n), \qquad \vartheta_3(q^2)\vartheta_2^2(q^2) = 4\Sigma q^m F(2m);$$
$$(43)\qquad m = 4k + 1: \qquad \vartheta_2(q^4)\vartheta_3^2(q^4) = 4\Sigma q^m F(m);$$
$$(44)\qquad m = 8k + 3: \qquad \vartheta_2^3(q^4) = 8\Sigma q^m F(m).$$

12. Using the series in (42), (40), the last after replacing q by $-q$, we find in the usual way from the identity $\vartheta_3 \times \vartheta_3^3 = \vartheta_3^4$ the class number relation

$$(45)\qquad 6H(n) + 12\Sigma H(n - a^2) = 4(-1)^n\lambda(n) - e(n).$$

To reverse this we proceed as in § 6, finding the H-reverse from the identity $\vartheta_3^3 = \vartheta_3^4 \times 1/\vartheta_3$ by means of (24), getting at once

$$(46)\qquad 12H(n) = 8(-1)^n\lambda(n) + D(n) + 8\Sigma(-1)^a\lambda(a)D(n - a).$$

13. Similarly from $\vartheta_2^3 \times \vartheta_2 = \vartheta_2^4$ we find by (44) the relation

$$(47)\qquad \Sigma F(4m - \mu^2) = \zeta(m);$$

and from $\vartheta_2^3 = \vartheta_2^4 \times 1/\vartheta_2$ by (32) the F-reverse of this,

$$(48)\qquad F(4m - 1) = \zeta(m) + \Sigma\zeta(\mu)T'\left(\frac{m - \mu}{2}\right),$$

which, on making an obvious change in notation, may be written more conveniently,

$$(49)\qquad m = 8k + 3: \qquad F(m) = \zeta\left(\frac{m + 1}{4}\right) + \Sigma\zeta(\mu)T'\left(\frac{m - 4\mu + 1}{8}\right).$$

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* Hermite, J. des Math., 1862, p. 25, and formulas (A), (B), (C) of Oeuvres, vol. 4, p. 138. The right of (C) is a misprint for $\vartheta_2^3(q)$. Notice his convention regarding F in formula (B); see footnote to § 14.

14. From $\vartheta_2\vartheta_3^2 \times \vartheta_2 = \vartheta_2^2\vartheta_3^2$ we have in the same way by (43), (41)

$$2\Sigma F(2m - \mu^2) = \zeta(m); \tag{50}$$

and from $2\sqrt[4]{q}\vartheta_2\vartheta_3^2 = \vartheta_2^2\vartheta_3^2 \times 2\sqrt[4]{q}/\vartheta_2$ on using (32), we find the F-reverse

$$2F(2m - 1) = \zeta(m) + \tfrac{1}{2}\Sigma[1 + (-1 \mid \mu m)]\zeta(\mu)T'\left(\frac{m - \mu}{4}\right). \tag{51}$$

The factor $\frac{1}{2}[1 + (-1 \mid \mu m)] = 1$ or 0 according as μ, m are congruent or incongruent modulo 4. In (50), (51), as always, F is taken with the usual conventions* for Kronecker's formulas.

15. As a last example we find the F-reverse of

$$F(2m) + 2\Sigma F(2m - 4a^2) = \zeta(m) \tag{52}$$

which comes from $\vartheta_2^2\vartheta_3 \times \vartheta_3 = \vartheta_2^2\vartheta_3^2$;

$$F(2m) = \zeta(m) + \Sigma\zeta(\mu)D\left(\frac{m - \mu}{2}\right). \tag{53}$$

By § 4 each of the pairs (45) and (46), (47) and (49), (50) and (51), (52) and (53) are formally equivalent in the sense that each member of a pair implies the other, and it is possible to transform each into the other arithmetically. Again, the first member in any pair is a reverse of the second member with respect to a certain function; thus (52) is the e_2-reverse of (53), where $e_2(n) = 1$ or 0 according as n is or is not the square of an even integer > 0. Each of the pairs may be reduced to several different forms by means of the elementary properties of F, F_1, G, H.

The other relations of the classical types involve what Hermite called incomplete functions instead of the complete functions ζ, ζ', λ, viz., functions of the divisors d, δ of n subject to inequalities, such as $d < \delta$. The reversion of such relations requires the definitions of several (incomplete) functions, but introduces no principle distinct from the preceding.

IV. RECURRENCES FOR THE TOTAL FUNCTIONS.

16. To state these we require but one more well-known function, $\xi(n)$, = the excess of the number of divisors of n that are $\equiv 1 \bmod 4$ over the number $\equiv 3 \bmod 4$, so that $\xi(4k + 3) = 0$, and $4\xi(n) =$ the number of representations of n as a sum of two squares whose roots are $\gtreqless 0$. The recurrences are derived by the same simple process as the class number

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* This is emphasized because there seems to be some confusion in Hermite's notation for the development of $\vartheta_2\vartheta_3^2$ (Oeuvres, vol. 4, pp. 138, 148). We must take $F(1)$, $F(9)$, $F(25)$, $F(49)$, $F(81)$, $\cdots = 1/2, 5/2, 5/2, 9/2, 17/2, \cdots$ in accord with the usual conventions and not, as Hermite appears to intend, $0, 2, 2, 4, 8, \cdots$. This may be verified by putting $m = 1, 5, 13, 25, 41, \cdots$ in (50).

relations and their reversions in the preceding section. Some are obvious from the definitions, others are less evident.

17. Multiplying both sides of (24) by ϑ_3 and equating coefficients of q^n, we get

$$D(n) + 2\Sigma D(n - a^2) = -2e(n); \tag{54}$$

and from (26) in the same way

$$N(n) - 2\Sigma N(n - a^2) = 2e(n); \tag{55}$$

whence, by adding and subtracting,

$$E(n) = 2\Sigma O(n - a^2), \qquad O(n) = 2\Sigma E(n - a^2) + 2e(n), \tag{56}$$

from which with the initial condition $E(1) = 0$, O, E, and hence O_1, O_2, E_1, E_2 may be rapidly calculated. It is advantageous in practice to separate the cases. Replacing E, O by $E_1 + E_2$, $O_1 + O_2$ respectively in (62), and combining, we have

$$\begin{aligned} E_1(m) &= 2\Sigma[O_1(m - 4a^2) + O_2(m - \mu^2)], \\ E_2(2n) &= 2\Sigma[O_1(2n - \mu^2) + O_2(2n - 4a^2)], \\ O_1(m) &= 2\Sigma[E_1(m - 4a^2) + E_2(m - \mu^2)] + 2e(m), \\ O_2(m) &= 2\Sigma[E_1(2n - \mu^2) + E_2(2n - 4a^2)] + 2e(2n), \end{aligned} \tag{57}$$

which, with the initial conditions

$$E_1(1) = 0, \qquad E_2(2) = 4, \qquad O_1(1) = 2, \qquad O_2(2) = 0,$$

suffice for the simultaneous computation by recurrence of the four functions. Obviously the suffixes in (57) may be suppressed.

18. We may eliminate O, E in turn from (56), getting thus recurrences involving O, E separately. On reduction of the results by means of the theorems for the representations of a number as a sum of two squares, these recurrences may be cast into forms involving single summations in place of the double introduced by the elimination. It is simpler, however, to derive these otherwise. We have $\vartheta_3^2 = 1 + 4\Sigma q^n \xi(n)$; hence from (28), proceeding as in § 17, we find

$$E(n) - 4\Sigma[\xi(a) - e(a)]E(n - a) = 4[\xi(n) - e(n)]; \tag{58}$$

and similarly from (29),

$$O(n) - 4\Sigma[\xi(a) - e(a)]O(n - a) = 2e(n); \tag{59}$$

whence, adding and subtracting, we have

$$N(n) - 4\Sigma[\xi(a) - e(a)]N(n - a) = 2[2\xi(n) - e(n)], \tag{60}$$

$$D(n) - 4\Sigma[\xi(a) - e(a)]D(n - a) = 2[2\xi(n) - 3e(n)]. \tag{61}$$

In using these we require the successive values of $\xi(n)$, which presuppose the resolution of 1, 2, $\cdots$, n into prime factors. To avoid this tentative process we find a recurrence for the computation of $\xi(n)$ from the identity $\vartheta_0 \times \vartheta_2\vartheta_3 = \vartheta_1'$:

$$(62) \quad m = 4k + 1: \quad \xi(m) + 2\Sigma(-1)^a\xi(m - 4a^2) = e(m)\sqrt{m}(-1 \mid \sqrt{m}),$$

which, with $\xi(4k + 3) = 0$ and $\xi(2^a m) = \xi(m)$, is sufficient for the non-tentative calculation of all the coefficients in (58)–(61). There is another recurrence for $\xi(n)$, but it is less simple than (62). Incidentally we note that (62) enables us to calculate the number of representations of an integer as a sum of two squares by recurrence. There are similar theorems for any *odd* number of squares up to 13.*

19. Passing to recurrences for the T functions we have from (32) on multiplying throughout by $\vartheta_2(q^4)$, and equating coefficients of like powers of q,

$$m = 8k + 1: \qquad \Sigma T'\left(\frac{m - \mu^2}{8}\right) = -e(m);$$

whence, by an obvious change in notation,

$$(63) \qquad T'(n) + \Sigma T'\left(\frac{8n + 1 - (\mu + 2)^2}{8}\right) = -e(8n + 1).$$

In the same way from (33),

$$(64) \qquad T(n) - \Sigma T\left(\frac{8n + 1 - (\mu + 2)^2}{8}\right) = e(8n + 1),$$

and combining (63), (64) we have the following for the calculation simultaneously of T_1, T_2,

$$(65) \qquad T_1(n) = e(8n + 1) + \Sigma T_2\left(\frac{8n + 1 - (\mu + 2)^2}{8}\right),$$

$$(66) \qquad T_2(n) = \Sigma T_1\left(\frac{8n + 1 - (\mu + 2)^2}{8}\right).$$

From (34), (35) we find recurrences for the separate calculation of T_1, T_2:

$$(67) \quad T_1(n) = e(8n + 1) + \Sigma[\xi(4a + 1) - 2e(8a + 1)]T_1(n - a),$$

$$(68) \quad \begin{aligned} T_2(n) = {} & \xi(4n + 1) - 2e(8n + 1) \\ & + \Sigma[\xi(4a + 1) - 2e(8a + 1)]T_2(n - a); \end{aligned}$$

whence, by addition and subtraction,

$$(69) \quad \begin{aligned} T(n) = {} & \xi(4n + 1) - e(8n + 1) \\ & + \Sigma[\xi(4a + 1) - 2e(8a + 1)]T(n - a), \end{aligned}$$

$$(70) \quad \begin{aligned} T'(n) = {} & \xi(4n + 1) - 3e(8n + 1) \\ & + \Sigma[\xi(4a + 1) - 2e(8a + 1)]T'(n - a). \end{aligned}$$

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* American Journal, July, 1920.

20. From (36)–(39) we similarly derive the following for the Ω-functions:

(71) $$\Omega(n) = [1 - (-1)^n]e(n) + 2\Sigma\Omega(n - \mu^2),$$

(72) $$\Omega'(n) = [(-1)^n - 1]e(n) - 2\Sigma\Omega'(n - \mu^2),$$

(73) $$\Omega_2(2n) = 2[1 - (-1)^n]\xi(n) + 4\Sigma\xi(\mu)\Omega_2(2n - 2\mu),$$

(74) $$\Omega_1(m) = 2e(m) + 4\Sigma\xi(\mu)\Omega_1(m - 2\mu).$$

21. The appended table with $F(11) = 3$, $F(19) = 3$, $F(27) = 4$, will be found sufficient for the numerical verification of all formulas in sections III, IV. In using the table we make the elementary transformations $E_2(2n) = E(2n)$, $N(n) = E(n) + O(n)$, etc., whenever necessary.

n	E	O	T_1	T_2	Ω	F	H	ξ	ζ	λ
1	0	2	1	0	2	1/2	1/2	1	1	− 1
2	4	0	0	1	4	1	1	1	3	3
3	0	8	2	0	8	1	2/3	0	4	− 4
4	16	2	0	3	16	1	$\frac{1}{2}$	1	7	3
5	8	32	4	0	32	2	2	2	6	− 6
6	64	24	1	6	64	2	2	0	12	12
7	64	128	9	2	128	1	0	0	8	− 8
8	260	160	3	13	256	2	1	1	15	3
9	384	538	19	6	514	5/2	5/2	1	13	−13
10	1128	896	12	28	1032	2	2	2	18	18

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